ON THE TOLMAN-OPPENHEIMER-VOLKOFF-DE SITTER EQUATION

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Abstract

Spherically symmetric static solutions of the Einstein equations with a positive cosmological constant for the energy-momentum tensor of a barotropic perfect fluid are governed by the Tolman-Oppenheimer-Volkoff-de Sitter equation. Sufficient conditions for existence of solutions with finite radii are given. The interior metric of the solution is connected with the Schwarzschild-de Sitter metric on the exterior vacuum region. The analytic property of the solutions at the vacuum boundary is investigated.

1 Introduction

We consider a static and spherically symmetric metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2F(r)}c^2dt^2 - e^{2H(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which satisfies the Einstein-de Sitter equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu},$$

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for the energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (c^2\rho + P)U^\mu U^\mu - Pg^{\mu\nu}.$$

Here $R_{\mu\nu}$ is the Ricci tensor associated with the metric $g_{\mu\nu}dx^\mu dx^\nu$, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature, and c, G are positive constants, the speed of light, the gravitational constant. Λ is the cosmological constant which is supposed to be positive. ρ is the mass density, P is the pressure and U^μ is the four-velocity. See [10, §111].

Historically speaking, the cosmological constant Λ of the above Einstein-de Sitter equations was introduced by A. Einstein [7], 1917, and was discussed soon by W. de Sitter [16]. Although it was introduced for a static universe, it was not necessary for an expanding universe. So, later Einstein wrote to H. Weyl "If there is no quasi-static world, then away with the cosmological term! " on May 23, 1923, and finally he rejected the cosmological term in [8], 1931, for the reason that it is not necessary to explain the Hubble's report on the redshifts of galaxies showing the expansion of the universe. (See [14, §15e].) However, although the original motivation for introducing the cosmological term disappeared, its status in the cosmological theories remained and it revived with new meanings in the recent development of the theories and observations. For the details, see the review [4]

For the static spherically symmetric metric, the Einstein-de Sitter equations are reduced to

$$\frac{dm}{dr} = 4\pi r^2\rho, \quad \frac{dP}{dr} = -(\rho + P/c^2)\frac{G\Big(m + \dfrac{4\pi r^3}{c^2}P\Big) - \dfrac{c^2\Lambda}{3}r^3}{r^2\Big(1 - \dfrac{2Gm}{c^2 r} - \dfrac{\Lambda}{3}r^2\Big)}. \tag{1}$$

The coefficients of the metric are given by

$$e^{2F(r)} = \kappa_+ e^{-2u(r)/c^2},$$

$u(r), \kappa_+$ being the function and the constant specified later, and

$$e^{-2H(r)} = 1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3}r^2.$$

When $\Lambda = 0$, the equations (1) turns out to be

$$\frac{dm}{dr} = 4\pi r^2\rho, \quad \frac{dP}{dr} = -(\rho + P/c^2)\frac{G\Big(m + \dfrac{4\pi r^3}{c^2}P\Big)}{r^2\Big(1 - \dfrac{2Gm}{c^2 r}\Big)}, \tag{2}$$

and this (2) is called the Tolman-Oppenheimer-Volkoff equation. It was derived in [13], 1939. Therefore we shall call (1) with $\Lambda > 0$ **the Tolman-Oppenheimer-Volkoff-de Sitter equation**. In this article we investigate this Tolman-Oppenheimer-Volkoff-de Sitter equation (1). Throughout this article we keep the following

Assumption *The pressure P is a given function of the density $\rho > 0$ such that $0 < P$ and $0 < dP/d\rho < c^2$ for $\rho > 0$ and $P \to 0$ as $\rho \to +0$. Moreover we assume that there are positive constants A, γ and an analytic function Ω on a neighborhood of $[0, +\infty)$ such that $\Omega(0) = 1$ and*

$$P = A\rho^\gamma \Omega(A\rho^{\gamma-1}/c^2).$$

We assume that $1 < \gamma < 2$.

Actually we are keeping in mind the equation of state for neutron stars:

$$P = Kc^5 \int_0^\zeta \frac{q^4 dq}{\sqrt{1+q^2}}, \quad \rho = 3Kc^3 \int_0^\zeta \sqrt{1+q^2}q^2 dq,$$

K being a positive constant. See [12], [17, p. 188].

The Tolman-Oppenheimer-Volkoff-de Sitter equation has already been systematically investigated from the physical point of view by C. G. Böhmer, [1], [2]. However in the study by C. G. Böhmer there is supposed to exist a positive density ρ_b, so called the 'boundary density', at which the pressure P vanishes, that is, $P > 0 \Leftrightarrow \rho > \rho_b$, and cosmological constants Λ satisfying $\Lambda < \dfrac{4\pi G}{c^2}\rho_b$ are considered. This situation is not treated in this article.

For the sake of notational conventions, we shall denote

$$\kappa(r, m) := 1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3}r^2, \tag{3}$$

and

$$Q(r, m, P) := G\Big(m + \frac{4\pi r^3}{c^2}P\Big) - \frac{c^2\Lambda}{3}r^3, \tag{4}$$

so that the second equation of (1) reads

$$\frac{dP}{dr} = -(\rho + P/c^2)\frac{Q(r, m, P)}{r^2\kappa(r, m)}.$$

We consider the equation (1) on the domain

$$\mathcal{D} = \{(r, m, P) | 0 < r, |m| < +\infty, 0 < \rho, 0 < \kappa(r, m)\}.$$

Now we have a solution germ $(m(r), P(r))$ at $r = +0$, given the central density $\rho_c > 0$ with $P_c := P(\rho_c)$, such that

$$m = \frac{4\pi}{3}\rho_c r^3 + O(r^5), \tag{5}$$
$$P = P_c - (\rho_c + P_c/c^2)\Big(4\pi G(\rho_c + 3P_c/c^2) - c^2\Lambda\Big)\frac{r^2}{6} + O(r^4)$$

as $r \to +0$. Proof is the same as that of [12, Proposition 1]. See [11, §2, pp. 57-58].

We are interested in the prolongation of the solution germ to the right as long as possible in the domain $\mathcal{D}$. Actually the prolongation is unique, since the right-hand sides of (1) are analytic functions of r, m, P in $\mathcal{D}$. See [6, Chap.1, Sec.5].

Especially we want to have a sufficient condition for that the prolongation turns out to be 'monotone-short' in the following sense:

Definition 1 *A solution $(m(r), P(r)), 0 < r < r_+$, of (1) is said to be* **monotone-short** *if $r_+ < \infty$, $dP/dr < 0$ for $0 < r < r_+$ and $P \to 0$ as $r \to r_+$ and if $\kappa_+ > 0$ and $Q_+ > 0$, where*

$$\kappa_+ := \lim_{r \to r_+ - 0} \kappa(r, m(r)) = 1 - \frac{2Gm_+}{c^2 r_+} - \frac{\Lambda}{3} r_+^2,$$
$$Q_+ := \lim_{r \to r_+ - 0} Q(r, m(r), P(r)) = Gm_+ - \frac{c^2\Lambda}{3} r_+^3,$$

with

$$m_+ := \lim_{r \to r_+ - 0} m(r).$$

Since the solution germ behaves as (5) as $r \to +0$, we assume

$$\Lambda < \frac{4\pi G}{c^2}(\rho_c + 3P_c/c^2) \tag{6}$$

in order that $dP/dr < 0$ at least for $0 < r \ll 1$.

Remark. If the condition (6) does not hold but the equality

$$\Lambda = \frac{4\pi G}{c^2}(\rho_c + 3P_c/c^2) \tag{7}$$

holds exactly, then the solution of (1), (5) turns out to be $(m(r), P(r)) = (4\pi\rho_c r^3/3,\ P_c), 0 < r < \sqrt{3/L}$, where $L := \dfrac{8\pi G}{c^2}\rho_c + \Lambda$. In this very special case, the metric is reduced to

$$ds^2 = c^2dt^2 - \Big(1 - \frac{L}{3}r^2\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

after a suitable change of the scale of t. The space at $t =$ Const. is isometric to the half of the compact 3-dimensional hypersphere with radius $\sqrt{3/L}$ embedded in the 4-dimensional Euclidean space $\mathbb{R}^4 = \{(\xi^1, \xi^2, \xi^3, \xi^4)\}$ through

$$\xi^1 = r\sin\theta\cos\phi, \quad \xi^2 = r\sin\theta\sin\phi, \quad \xi^3 = r\cos\theta, \quad \xi^4 = \sqrt{\frac{3}{L} - r^2}.$$

Thus the horizon $r = \sqrt{3/L} - 0$ is merely apparent. Of course the condition (7) is highly unstable. This is nothing but the 'Einstein's steady state inverse (1917)' proposed in [7]. On the other hand, suppose that the inequality

$$\Lambda > \frac{4\pi G}{c^2}(\rho_c + 3P_c/c^2) \tag{8}$$

holds. Then the solution germ satisfies $dP/dr > 0$ for $0 < r \ll 1$. However it is possible that dP/dr become negative when prolonged to the right. This fact will be shown later. $\square$

We introduce the variable u by

$$u := \int_0^\rho \frac{dP}{\rho + P/c^2}.$$

Then we see

$$\begin{aligned}
u &= \frac{\gamma A}{\gamma - 1}\rho^{\gamma-1}\Omega_u(A\rho^{\gamma-1}/c^2), \\
\rho &= A_1 u^{\frac{1}{\gamma-1}}\Omega_\rho(u/c^2), \\
P &= AA_1^\gamma u^{\frac{\gamma}{\gamma-1}}\Omega_P(u/c^2),
\end{aligned}$$

where $\Omega_u, \Omega_\rho, \Omega_P$ are analytic functions on a neighborhood of $[0, +\infty[$ such that $\Omega_u(0) = \Omega_\rho(0) = \Omega_P(0) = 1$ and$A_1 := \Big(\dfrac{\gamma-1}{\gamma A}\Big)^{\frac{1}{\gamma-1}}$. The functions $\Omega_u, \Omega_\rho, \Omega_P$ are depending upon only γ and the function Ω. In fact we take

$$\Omega_u(\zeta) = \frac{1}{\zeta}\int_0^\zeta \frac{\Omega(\zeta') + \frac{\gamma-1}{\gamma}\zeta' D\Omega(\zeta')}{1+\zeta'\Omega(\zeta')}d\zeta',$$
$$\zeta = \frac{\gamma-1}{\gamma}\eta\Omega_\rho(\eta) \Leftrightarrow \eta = \frac{\gamma}{\gamma-1}\zeta\Omega_u(\zeta),$$
$$\Omega_P(\eta) = \Omega(\zeta)\Omega_u(\zeta)^{-\frac{\gamma}{\gamma-1}} \quad \text{with} \quad \zeta = \frac{\gamma-1}{\gamma}\eta\Omega_\rho(\eta)$$

Let us fix a small positive number δ_Ω such that these functions are defined and analytic on a neighborhood of $[-\delta_\Omega, +\infty[$. We put

$$u_c := \int_0^{\rho_c} \frac{dP}{\rho + P/c^2} = \frac{\gamma A}{\gamma-1}\rho_c^{\gamma-1}\Omega_u(A\rho_c^{\gamma-1}/c^2).$$

2 Main result

We claim

Theorem 1 *Suppose that $6/5 < \gamma < 2$. Then there exists a positive number $\epsilon_0 (\leq 1)$ depending upon only γ and the function Ω such that if*

$$(9) \qquad u_c \leq c^2\epsilon_0, \qquad \Lambda \leq \frac{4\pi}{c^2}G\Big(\frac{\gamma-1}{\gamma A}\Big)^{\frac{1}{\gamma-1}}(u_c)^{\frac{1}{\gamma-1}}\epsilon_0,$$

then the prolongation of the solution germ with (5) to the right turns out to be monotone-short.

This can be considered as the de Sitter version of the result by A. D. Rendall and G. B. Schmidt, [15].

Let us sketch the proof. Using the variable u, we can write the Tolman-Oppenheimer-Volkoff-de Sitter equation (1) as

$$(10) \qquad \frac{dm}{dr} = 4\pi r^2 A_1 (u_\sharp)^{\frac{1}{\gamma-1}}\Omega_\rho(u/c^2),$$
$$\frac{du}{dr} = -\frac{G\Big(m + \frac{4\pi}{c^2}r^3 A A_1^\gamma (u_\sharp)^{\frac{\gamma}{\gamma-1}}\Omega_P(u/c^2)\Big) - \frac{c^2\Lambda}{3}r^3}{r^2\Big(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3}r^2\Big)}.$$

Here $(u_\sharp)$ stands for $\max\{u,0\}$. Since we are assuming that $1<\gamma<2$, which implies $\mu := \dfrac{1}{\gamma-1} > 1, \dfrac{\gamma}{\gamma-1} = \mu+1 > 2$, we can see that the functions $u \mapsto (u_\sharp)^\mu, (u_\sharp)^{\mu+1}$ are of class $C^1(\mathbb{R})$. Keeping in mind it, we consider that the domain of the equation(10) is

$$\mathcal{D}_u = \{(r,m,u) | 0 < r, |m| < \infty, -\delta_\Omega < u/c^2 < +\infty, \kappa > 0\}.$$

Let us perform the homologous transformation of the variables

$$r = aR, \qquad m = a^3 b^{\frac{1}{\gamma-1}} \cdot 4\pi A_1 M, \qquad u = bU,$$

where a, b are positive parameters. We take $b = u_c$ and a which satisfies

$$4\pi G A_1 a^2 b^{\frac{2-\gamma}{\gamma-1}} = 1.$$

Let us write

$$\lambda := \frac{c^2}{4\pi G A_1}\Lambda, \qquad \alpha := b/c^2 = u_c/c^2, \qquad \beta := b^{-\frac{1}{\gamma-1}}\lambda = \frac{c^2}{4\pi G A_1}(u_c)^{-\frac{1}{\gamma-1}}\Lambda.$$

Then the system (10) turns out to

$$\text{(11)} \qquad \frac{dM}{dR} = R^2 (U_\sharp)^{\frac{1}{\gamma-1}} \Omega_\rho(\alpha U),$$

$$\frac{dU}{dR} = -\frac{1}{R^2} \frac{\Big(M + \dfrac{\gamma-1}{\gamma}\alpha R^3 (U_\sharp)^{\frac{\gamma}{\gamma-1}} \Omega_P(\alpha U) - \dfrac{1}{3}\beta R^3\Big)}{\Big(1 - 2\alpha \dfrac{M}{R} - \dfrac{1}{3}\alpha\beta R^2\Big)}.$$

Here $(U_\sharp)$ stands for $\max\{U,0\}$. The domain of the system (11) should be

$$\mathcal{D}_U = \{(R,M,U) | 0 < R, |M| < \infty, -\delta_\Omega < U < 2, \kappa > 0\},$$

where, of course,

$$\kappa = 1 - 2\alpha\frac{M}{R} - \frac{1}{3}\alpha\beta R^2.$$

Let us concentrate ourselves to $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$. We are considering a solution germ $(M(R), U(R))$ at $R = +0$ which satisfies

$$\text{(12)} \qquad M(R) = \Omega_\rho(\alpha)\frac{R^3}{3} + O(R^5),$$

$$U(R) = 1 - \Big(\Omega_\rho(\alpha) + \frac{3\gamma}{\gamma-1}\alpha\Omega_P(\alpha) - \beta\Big)\frac{R^2}{6} + O(R^4).$$

We claim

Proposition 1 *There is a positive number R_0 which depends upon only γ and the function Ω such that $(M(R), U(R))$ exists and satisfies $dU/dR < 0$ on $0 < R \leq R_0$ and $(M(R_0), U(R_0))$ depends continuously on $\alpha, \beta \in [0, 1]$.*

Proof is standard, and done by converting the system of differential equations (11) to a system of integral equations under the condition (12) as

(13)

$$q(R) = \frac{3}{R^3}\int_0^R U(R')^{\frac{1}{\gamma-1}}\frac{\Omega_\rho(\alpha U(R'))}{\Omega_\rho(\alpha)}R'^2 dR',$$

$$U(R) = 1 - \int_0^R \frac{\frac{1}{3}\Omega_\rho(\alpha)q(R') + \frac{\gamma-1}{\gamma}\alpha U(R')^{\frac{\gamma}{\gamma-1}}\Omega_P(\alpha U(R')) - \frac{1}{3}\beta}{1 - 2\alpha\frac{1}{3}\Omega_\rho(\alpha)q(R')R'^2 - \frac{1}{3}\alpha\beta R'^2}R' dR'.$$

Then, taking δ sufficiently small uniformly on α, β, we see that the mapping $(q, U) \mapsto (\tilde{q}, \tilde{U})$, which is the right-hand side of the (13), is a contraction from $\mathfrak{F} = \{(q, U) \in C[0, \delta] | 0 \leq q \leq C_q, \frac{1}{2} \leq U \leq 2\}$ into itself with respect to a suitable functional distance, where

$$C_q := \max\Big\{U^{\frac{1}{\gamma-1}}\frac{\Omega_\rho(\alpha U)}{\Omega_\rho(\alpha)} \ \Big|\ \frac{1}{2} \leq U \leq 2, \quad 0 \leq \alpha \leq 1\Big\}.$$

For the details see [11, pp. 57-58].

Let us come back to the proof of Theorem 1. The right-hand side of the system (11) depends continuously on α, β, and tends to $(R^2(U_\sharp)^{\frac{1}{\gamma-1}}, -M/R^2)^T$ as $\alpha \to 0, \beta \to 0$. The limit system

$$\frac{dM}{dR} = R^2(U_\sharp)^\mu, \qquad \frac{dU}{dR} = -\frac{M}{R^2}$$

is nothing but the Lane-Emden equation

$$-\frac{1}{R^2}\frac{d}{dR}\Big(R^2\frac{dU}{dR}\Big) = (U_\sharp)^\mu. \tag{14}$$

Since we are assuming that $6/5 < \gamma < 2$, say, $1 < \mu < 5$, the solution $U = \bar{U}(R)$ with $\bar{U}(0) = 1$ of the Lane-Emden equation (14) is short, that is, $0 < \bar{U}(R), d\bar{U}/dR < 0$ for $0 < R < \xi_1 = \xi_1(\gamma)$ and $\bar{U}(\xi_1) = 0$. See [5], [9]. Of course we consider

$$\bar{U}(R) = \Big(R^2\frac{d\bar{U}}{dR}\Big)_{R=\xi_1}\Big(\frac{1}{\xi_1} - \frac{1}{R}\Big)$$

harmonically on $R \geq \xi_1$.

Thanks to Proposition 1, if ϵ_0 is sufficiently small and if

$$\alpha \leq \epsilon_0 \qquad \text{and} \qquad \beta \leq \epsilon_0, \tag{15}$$

then $U = U(R)$ exists and remains near to the orbit of $U = \bar{U}(R)$ on $R_0 \leq R \leq \xi_1 + \delta_R$, δ_R being small so that

$$-\frac{\delta_\Omega}{2} \leq \bar{U}(\xi_1 + \delta_R) < 0.$$

This is nothing but a direct application of [6, Theorem 7.4]. Note that $\max\{\Omega_P(\eta) | -\delta_\Omega \leq \eta \leq 2\}$ depends upon only γ and the function Ω, and we have $-\delta_\Omega \leq \eta = \alpha U \leq 2$ provided that $\alpha \leq \epsilon_0 \leq 1$ and $-\delta_\Omega < U < 2$. Especially if $U(\xi_1 + \delta_R) < 0$, then the radius R_+ of $U(R)$ should be found in the interval $]0, \xi_1 + \delta_R[$. This completes the proof of Theorem 1, since the condition (15) is nothing but (9).

Note that (6) follows from (9) if ϵ_0 is sufficiently small, since

$$\rho + 3P/c^2 = \Big(\frac{\gamma - 1}{\gamma A}\Big)^{\frac{1}{\gamma-1}} u^{\frac{1}{\gamma-1}} \Omega_{\rho+3P/c^2}(u/c^2),$$

where

$$\Omega_{\rho+3P/c^2}(\eta) := \Omega_\rho(\eta) + 3\frac{\gamma-1}{\gamma}\eta\Omega_P(\eta)$$

is a function depending upon only γ and the function Ω and $\Omega_{\rho+3P/c^2}(0) = 1$ so that we can assume that $\min\{\Omega_{\rho+3P/c^2}(\eta) | -\delta_\Omega \leq \eta \leq 2\} > \epsilon_0$.

Remark. For the existence of u_c satisfying (9) it is necessary that Λ enjoys

$$\Lambda \leq 4\pi c^{\frac{2(2-\gamma)}{\gamma-1}} G\Big(\frac{\gamma-1}{\gamma A}\Big)^{\frac{1}{\gamma-1}} \epsilon_0^{\frac{\gamma}{\gamma-1}}.$$

Thus one may ask whether the real value of the cosmological constant of our universe satisfies it or not. But this question is not theoretical but experimental-observational and numerical. To answer to it is not a business of such a poor mathematician as the author of this article. □

Even if $\gamma \leq 6/5$, the solution with the central density ρ_c of the Tolman-Oppenheimer-Volkoff equation, that is, (1) with $\Lambda = 0$, or (2), can be short,

if ρ_c is large and the function $P(\rho)$ is very much different from the exact γ-law for large ρ. For a sufficient condition for solutions to be short, see [12, Proposition 3], the proof of [11, Theorem 1]. Therefore we can consider such a case, supposing that $1 < \gamma < 2$ and the solution $(m, P) = (m^0(r), P^0(r)), 0 < r < r^0_+$, of (2) with the same central density ρ_c satisfies $P^0(r) \to 0$ as $r \to r^0_+ - 0$, with r^0_+ being finite. Then the associated $(m, u) = (m^0(r), u^0(r)), 0 < r < r^0_+$, satisfies (10) with $\Lambda = 0$, that is,

$$\frac{dm}{dr} = 4\pi r^2 A_1 (u_\sharp)^{\frac{1}{\gamma-1}} \Omega_\rho(u/c^2), \tag{16}$$
$$\frac{du}{dr} = -\frac{G\Big(m + \dfrac{4\pi}{c^2} r^3 A A_1^\gamma (u_\sharp)^{\frac{\gamma}{\gamma-1}} \Omega_P(u/c^2)\Big)}{r^2\Big(1 - \dfrac{2Gm}{c^2 r}\Big)}.$$

In order to extend $(m^0(r), u^0(r))$ onto $r \geq r^0_+$, we put

$$m^0(r) = m^0_+ (:= m^0(r^0_+)),$$
$$u^0(r) = \frac{c^2}{2}\Big(\log\Big(1 - \frac{2Gm^0_+}{c^2 r^0_+}\Big) - \log\Big(1 - \frac{2Gm^0_+}{c^2 r}\Big)\Big),$$

for $r \geq r^0_+$. Then the extended $(m^0(r), u^0(r))$ satisfies (16) on $0 < r < +\infty$. Since the right-hand side of (10) tends to that of (16) as $\Lambda \to 0$, the solution of (10) under consideration exists and remains in a neighborhood of the orbit $(m^0(r), u^0(r))$ on $0 < r \leq r^0_+ + \delta_r$, δ_r being a sufficiently small positive number, provided that Λ is sufficiently small. Thus we have

Theorem 2 *Suppose that the solution of the Tolman-Oppenheimer-Volkaff equation (2) with central density ρ_c is short. Then there exists a small positive number ϵ_1 such that, if $\Lambda \leq \epsilon_1$, the solution germ of the Tolman-Oppenheimer-Volkoff-de Sitter equation (1) with the central density ρ_c has a monotone-short prolongation.*

We should note that ϵ_1 may depend not only upon γ and the function Ω but also upon A, c, G and ρ_c. In contrast with Theorem 1, we have no hope to specify the manner of dependence.

3 Monotonicity

Here let us give a remark on the monotonicity of the solution of the Tolman-Oppenheimer-Volkoff-de Sitter equation (1).

When we studied the Tolman-Oppenheimer-Volkoff equation, that is, (1) with $\Lambda = 0$, or (2), we see that if $]0, r_+[, r_+ \leq +\infty$, is the right maximal interval of existence of the solution in the domain $\mathcal{D}$, then $dP/dr < 0$ for $0 < r < r_+$ and $P \to 0$ as $r \to r_+ - 0$. Proof is given in [12], [11]. In other words, we can say on the Tolman-Oppenheimer-Volkoff equation that, if the prolongation of the solution germ under consideration is short, it is necessarily monotone-short. However it is not the case on the Tolman-Oppenheimer-Volkoff-de Sitter equation with $\Lambda > 0$. Even if $dP/dr < 0$ for $0 < r \ll 1$ under the assumption (6), dP/dr may turn out to be positive during the prolongation. Let us show it.

In order to fix the idea, we suppose $u_c = 1$, and put

$$r = aR, \quad m = a^3 \cdot 4\pi A_1 M, \quad 4\pi G A_1 a^2 = 1, \quad u = U, \quad \lambda = \frac{c^2}{4\pi G A_1}\Lambda.$$

Then the system (1) is reduced to

$$\begin{aligned}
&\frac{dM}{dR} = R^2 U^\mu \Omega_\rho(U/c^2), \\
&\frac{dU}{dR} = -\frac{1}{R^2}\Big(M + \frac{\gamma-1}{\gamma}\frac{R^3}{c^2}U^{\mu+1}\Omega_P(U/c^2) - \frac{\lambda}{3}R^3\Big)\times \\
&\qquad \times \Big(1 - \frac{2M}{c^2 R} - \frac{\lambda}{3c^2}R^2\Big)^{-1}.
\end{aligned} \tag{17}$$

Here $\mu := 1/(\gamma - 1)$.

The right-hand side of the system (17) depends continuously upon the speed of light c and tends to $\Big(R^2U^\mu, -\frac{1}{R^2}\Big(M - \frac{\lambda}{3}R^3\Big)\Big)^T$ as $c \to \infty$. This non-relativistic limit equation can be written as

$$-\frac{1}{R^2}\frac{d}{dR}\Big(R^2\frac{dU}{dR}\Big) = U^\mu - \lambda. \tag{18}$$

In this situation we are assuming that Λ depends upon c and $c^2\Lambda/(4\pi G A_1)$ tends to λ. Since (18) is the Lane-Emden equation when $\lambda = 0$, we shall call it '**the Lane-Emden-de Sitter equation**' supposing that $\lambda > 0$.

Although we are supposing $1 < \mu < +\infty (\Leftrightarrow 1 < \gamma < 2)$, we observe the limiting case $\mu = 1 (\Leftrightarrow \gamma = 2)$. Then the equation (18) is linear and the solution $U = \hat{U}(R)$ with $\hat{U}(0) = 1$ is given by

$$\hat{U}(R) = \lambda + (1 - \lambda)\frac{\sin R}{R}$$

explicitly. We have

$$\hat{U}(R) = 1 - \frac{1-\lambda}{6}R^2 + O(R^4)$$

as $R \to +0$, and the condition (6) reads $\lambda < 1$. Suppose that $\frac{1}{2} \leq \lambda < 1$. Then we find that

$$\frac{d\hat{U}}{dR} = \frac{1-\lambda}{R}\Big(\cos R - \frac{\sin R}{R}\Big)$$

turns out to be positive for $3\pi/2 < R < 2\pi$ and so on, while $\hat{U}(R) > 0$ exists and oscillates on $0 < R < +\infty$, and converges to λ as $R \to +\infty$. Therefore, this explicit example tells us that, if γ is near to 2, c is sufficiently large, and $c^2\Lambda/(4\pi G A_1)$ is near to a number λ in the interval $[\frac{1}{2}, 1[$, the behavior of the solution under consideration may be similar, that is, the prolongation of the solution germ with $u_c = 1$ is not monotone. On the other hand, the condition (8) reads $\lambda > 1$. Then $d\hat{U}/dR > 0$ for $0 < R \ll 1$ but $d\hat{U}/dR$ become negative and $\hat{U}(R)$ oscillates and tends to the limit λ as $R \to +\infty$. Therefore, this explicit example tells us that, if γ is near to 2, c is sufficiently large, and $c^2\Lambda/(4\pi G A_1)$ is near to $\lambda > 1$, then $P(r)$ of the prolongation of the germ, which satisfies $dP/dr > 0$ as $0 < r \ll 1$, become decreasing.

In the definition of 'monotone-short' solutions of the Tolman-Oppenheimer-Volkoff-de Sitter equation we have required that $\kappa_+ > 0$ and $Q_+ > 0$. Let us spend few words concerning these conditions.

Consider a solution $(m, P) = (m(r), P(r)), 0 < r < r_+$, in $\mathcal{D}$ such that $dP/dr < 0$, which requires $Q(r, m(r), P(r)) > 0$ for $0 < r < r_+$, and suppose $P(r) \to 0$ as $r \to r_+ - 0$, with r_+ being finite.

When we are concerned with the Tolman-Oppenheimer-Volkoff equation (2) with $\Lambda = 0$, the condition $\kappa_+ > 0$ follows automatically. Proof can be found in [12]. Of course, if $\Lambda = 0$, then $Q_+ = Gm_+ > 0$ a priori.

However if $\Lambda > 0$ it seems that we cannot exclude the possibility that $\kappa_+ = 0$ a priori. Generally speaking, since $\kappa > 0$ in $\mathcal{D}$, we have $\kappa_+ \geq 0$. Suppose $\kappa_+ = 0$. Then

$$\frac{2Gm_+}{c^2 r_+} = 1 - \frac{\Lambda}{3}r_+^2,$$

and, since $\kappa > 0$ for $r < r_+$ and $\kappa_+ = 0$, we see

$$\begin{aligned} \kappa'_+ := \frac{d\kappa}{dr}\Big|_{r=r_+-0} &= \lim_{r\to r_+-0} -\frac{2G}{c^2}4\pi\rho + \frac{2Gm}{c^2}\frac{1}{r^2} - \frac{2}{3}\Lambda r \\ &= \frac{2Gm_+}{c^2}\frac{1}{r_+^2} - \frac{2}{3}\Lambda r_+ = \frac{1}{r_+}(1-\Lambda r_+^2) \leq 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Q_+ &= Gm_+ - \frac{c^2\Lambda}{3}r_+^3 = \frac{c^2 r_+}{2}(1-\Lambda r_+^2) \\ &\geq 0, \end{aligned}$$

since $Q > 0$ for $r < r_+$. Thus it should be the case that $1 - \Lambda r_+^2 = 0$ and $Q_+ = 0$. In other words, $\kappa_+ = 0$ requires $Q_+ = 0$ and $\Lambda r_+^2 = 1$. This is very non-generic situation probably hard to occur, but at the moment we have no reason to exclude this possibility.

4 Metric on the vacuum region

Suppose that we have fixed a solution $(m(r), P(r)), 0 < r < r_+$, of the Tolman-Oppenheimer-Volkoff-de Sitter equation (1) which is monotone-short. Then we have the metric

$$ds^2 = \kappa_+ e^{-2u/c^2}c^2dt^2 - \frac{1}{\kappa}dr^2 - r^2d\omega^2$$

on $0 \leq r < r_+$, where

$$d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

We should continue this metric to the exterior vacuum domain $r \geq r_+$. Naturally, keeping in mind the Birkhoff theorem, we should take the Schwarzschild-de Sitter metric

$$ds^2 = \Big(1 - \frac{2Gm_+}{c^2r} - \frac{\Lambda}{3}r^2\Big)c^2dt^2 - \Big(1 - \frac{2Gm_+}{c^2r} - \frac{\Lambda}{3}r^2\Big)^{-1}dr^2 - r^2d\omega^2$$

on $r \geq r_+$. As the whole we take

$$ds^2 = g_{00}c^2dt^2 - g_{11}dr^2 - r^2d\omega^2,$$

where

$$g_{00} = \begin{cases} \kappa_+ e^{-2u(r)/c^2} & (0 \le r < r_+) \\ 1 - \dfrac{2Gm_+}{c^2 r} - \dfrac{\Lambda}{3} r^2 & (r_+ \le r < r_E) \end{cases},$$

$$-g_{11} = \Big(1 - \frac{2G\tilde{m}(r)}{c^2 r} - \frac{\Lambda}{3} r^2\Big)^{-1} \quad (0 \le r < r_E),$$

with

$$\tilde{m}(r) = \begin{cases} m(r) & (0 \le r < r_+) \\ m_+ & (r_+ \le r < r_E). \end{cases}$$

Here the constants $r_E, r_I, (0 < r_I < r_E < +\infty)$, are the values of r of the so called 'cosmological horizon', 'black hole horizon', that is, $\kappa(r, m_+) > 0$ if and only if $r_I < r < r_E$. In other words, we have

$$\kappa(r, m_+) = \frac{\Lambda}{3r}(r - r_I)(r_E - r)(r + r_I + r_E).$$

See [3]. But this situation is possible only if

$$\sqrt{\Lambda} < \frac{c^2}{3Gm_+}. \tag{19}$$

If (19) does not hold, then $\kappa(r, m_+) \le 0$ for all $r > 0$. However, since we are supposing $\kappa_+ = \kappa(r_+, m_+) > 0$, the condition (19) is supposed to hold and we have $r_I < r_+ < r_E$. Let us discuss the regularity of this patched metric.

First we observe the regularity of $u(r)$. We claim

Proposition 2 *The function $u(r)$ is of class C^2 in a neighborhood of r_+ and*

$$u(r) = B(r_+ - r)(1 + O(r_+ - r))$$

as $r \to r_+ - 0$, with $B := Q_+/r_+^2\kappa_+$. Hence $\rho(r)$ is of class C^1 and

$$\rho(r) = \Big(\frac{(\gamma - 1)B}{\gamma A}\Big)^{\frac{1}{\gamma-1}} (r_+ - r)^{\frac{1}{\gamma-1}} (1 + O(r_+ - r)).$$

Proof. Since $u(r)$ satisfies the equation (10), whose right-hand side is a C^1-function of (r, m, u) near $(r_+, m_+, 0)$ thanks to $\mu = \dfrac{1}{\gamma - 1} > 1$. Recall

that $\kappa_+ > 0$. Therefore the continuous solution $u(r)$ turns out to be of class C^2 and

$$\frac{du}{dr}\Big|_{r=r_+-0} = -\frac{Q+}{r_+^2\kappa_+} = -B.$$

This completes the proof.

Now we are going to see the regularity of g_{00}, g_{11}. Since

$$\frac{d}{dr}\tilde{m}(r) = \begin{cases} 4\pi r^2\rho(r) & (r < r_+) \\ 0 & (r_+ \leq r < r_E) \end{cases}$$

is of class C^1, $\tilde{m}(r)$ is of class C^2. Therefore g_{11} is twice continuously differentiable across $r = r_+$.

Since u vanishes at $r = r_+ - 0$, g_{00} is continuous thanks to the definition of κ_+. We see

$$\frac{d}{dr}g_{00}\Big|_{r=r_+-0} = -\frac{2\kappa_+}{c^2}\frac{du}{dr}\Big|_{r=r_+-0} = \frac{2Q_+}{c^2r_+^2}$$

and

$$\frac{d}{dr}g_{00}\Big|_{r=r_++0} = \Big(\frac{2Gm_+}{c^2r^2} - \frac{2\Lambda}{3}r\Big)_{r=r_+} = \frac{2Q_+}{c^2r_+^2}.$$

Therefore g_{00} is continuously differentiable. We have

$$\frac{d^2}{dr^2}g_{00}\Big|_{r=r_+-0} = \frac{4\kappa_+}{c^4}\Big(\frac{du}{dr}\Big)^2_{r=r_+-0} - \frac{2\kappa_+}{c^2}\Big(\frac{d^2u}{dr^2}\Big)_{r=r_+-0}.$$

But by a tedious calculation we have

$$\frac{d^2u}{dr^2}\Big|_{r=r_+-0} = \frac{c^2\Lambda}{\kappa_+} + \frac{2Q_+}{r_+^3\kappa_+} + \frac{2(Q_+)^2}{c^2r_+^4\kappa_+^2}.$$

This can be derived by differentiating the right-hand side of the equation for du/dr, that is, the second equation of (10). Therefore we see

$$\frac{d^2}{dr^2}g_{00}\Big|_{r=r_+-0} = \frac{d^2}{dr^2}g_{00}\Big|_{r=r_++0} = -\frac{4Q_+}{c^2r_+^3} - 2\Lambda.$$

Hence g_{00} is twice continuously differentiable across $r = r_+$. Summing up, we have

Theorem 3 *Given a monotone-short solution of the Tolman-Oppenheimer-Volkoff-de Sitter equation (1), we can extend the interior metric to the exterior Schwarzschild-de Sitter metric on the vacuum region with twice continuous differentiability.*

5 Analytical property of the vacuum boundary

Let us observe the analytical property of a monotone-short solution $(m(r), P(r))$, $0 < r < r_+$, of the Tolman-Oppenheimer-Volkoff-de Sitter equation (1). Proposition 2 tells us that the associated $u(r)$ belongs to $C^2([0, r_+])$ and

$$u(r) = B(r_+ - r)(1 + O(r_+ - r))$$

as $r \to r_+ - 0$, where $B = Q_+/r_+^2\kappa_+$. Moreover we claim

Theorem 4 *Any monotone-short solution $u(r), 0 < r < r_+$, of (1) enjoys the behavior at $r = r_+ - 0$ such that*

$$u(r) = B(r_+ - r)(1 + [r_+ - r, (r_+ - r)^{\frac{\gamma}{\gamma-1}}]_1),$$

therefore

$$\rho(r) = \Big(\frac{(\gamma-1)B}{\gamma A}\Big)^{\frac{1}{\gamma-1}}(r_+ - r)^{\frac{1}{\gamma-1}}(1 + [r_+ - r, (r_+ - r)^{\frac{\gamma}{\gamma-1}}]_1).$$

Here $[X_1, X_2]_1$ stands for a convergent double power series of the form

$$\sum_{k_1+k_2\geq 1} a_{k_1k_2}X_1^{k_1}X_2^{k_2}.$$

Proof. Let us denote $\mu := \dfrac{1}{\gamma-1}$ so that $\dfrac{\gamma}{\gamma-1} = \mu+1$.

First suppose that μ is an integer. Then proof is easy. In fact $(m(r), u(r))$ satisfies at least on $0 < r < r_+$ the system of equations

$$\frac{dm}{dr} = 4\pi r^2 A_1 u^\mu \Omega_\rho(u/c^2), \tag{20}$$

$$\frac{du}{dr} = -\frac{G\Big(m + \dfrac{4\pi}{c^2}r^3 AA_1^\gamma u^{\mu+1}\Omega_P(u/c^2)\Big) - \dfrac{c^2\Lambda}{3}r^3}{r^2\Big(1 - \dfrac{2Gm}{c^2r} - \dfrac{\Lambda}{3}r^2\Big)}.$$

and $(m(r), u(r)) \to (m_+, 0)$ as $r \to r_+ - 0$. But, since μ is supposed to be an integer, the right-hand side of the system (20) is analytic function of (r, m, u)

in a neighborhood of $(r_+, m_+, 0)$. This guarantees that $m(r), u(r)$ admit analytic prolongations beyond $r = r_+$ to the right, and completes the proof. Of course this analytic prolongation is different from the C^2-prolongation as a solution of (10), since $u^\mu \neq (u_\sharp)^\mu (= 0)$ for $u < 0$.

Now suppose that μ is not an integer. Since $u(r), 0 < r < r_+$, is monotone decreasing, it has the inverse function $r = r(u)$ defined on $0 < u < u_c$ such that $r(u) \to r_+$ as $u \to +0$. Then we have a solution $(m, r) = (m(u), r(u))$ of the system of equations

$$
\begin{aligned}
&\text{(21a)} \qquad \frac{dm}{du} = -4\pi r^4 \Big(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2\Big) \cdot Q^{-1} \cdot A_1 u^\mu \Omega_\rho(u/c^2), \\
&\text{(21b)} \qquad \frac{dr}{du} = -r^2 \Big(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2\Big) \cdot Q^{-1},
\end{aligned}
$$

where

$$Q = G\Big(m + \frac{4\pi}{c^2} r^3 A A_1^\gamma u^{\mu+1} \Omega_P(u/c^2)\Big) - \frac{c^2\Lambda}{3} r^3.$$

Since $(m(u), r(u)) \to (m_+, r_+)$ and $Q \to Q_+ > 0$ as $u \to +0$ and the right-hand sides of (21a)(21b) are analytic functions of u, u^μ, m, r on a neighborhood of $(0, 0, m_+, r_+)$, we can apply the following Lemma in order to get

$$
\begin{aligned}
m(u) &= m_+ + u[u, u^\mu]_0, \\
r(u) &= r_+ + u[u, u^\mu]_0.
\end{aligned}
$$

Here $[\cdot, \cdot]_0$ denotes a convergent double power series.

Lemma 1 *Let $\mu > 1$ and $f^\alpha(x, x^\mu, y_1, y_2), \alpha = 1, 2,$ be analytic functions of x, x^μ, y_1, y_2 on a neighborhood of $(0, 0, 0, 0)$. Let $(y_1(x), y_2(x)), 0 \le x \le \delta$, be the solution of the problem*

$$\text{(22)} \qquad \frac{dy_\alpha}{dx} = f^\alpha(x, x^\mu, y_1, y_2), \quad y_\alpha|_{x=0} = 0, \qquad \alpha = 1, 2.$$

Then there are analytic functions φ^α of x, x^μ on a neighborhood of $(0, 0)$ such that $y_\alpha(x) = x\varphi^\alpha(x, x^\mu)$ for $0 < x \ll 1$.

A proof of this Lemma will be sketched in the Appendix.

Since $dm/du \sim -Cu^\mu$, with $C = 4\pi r_+^4(\kappa_+/Q_+)A_1$, and $du/dr \to -B$, we have

$$m = m_+ - Cu^{\mu+1} + \sum_{n\geq 2} m_{1n}u^{\mu n+1} + \sum_{n\geq 0, l\geq 2} m_{ln}u^{\mu n+l},$$

$$r = r_+ - \frac{1}{B}u + \sum_{n\geq 1} c_{1n}u^{\mu n+1} + \sum_{n\geq 0, l\geq 2} c_{ln}u^{\mu n+l}.$$

If $m_{1n} \neq 0$ for $\exists n \geq 2$, then dm/du would contain the term $u^{\mu n}$ with $n \geq 2$. However it is impossible, since the right-hand side of (21a) cannot contain such a term. Therefore $m_{1n} = 0$ for $\forall n \geq 2$. If $c_{1n} \neq 0$ for $\exists n \geq 1$, then dr/du would contain the term $u^{\mu n}$. However it is impossible, since the right-hand side of the equation (21b) cannot contain such a term. Therefore $c_{1n} = 0$ for $\forall n \geq 1$. Thus we have

$$m = m_+ - Cu^{\mu+1} + \sum_{n\geq 0, l\geq 2} m_{ln}u^{\mu n+l},$$

$$r = r_+ - \frac{1}{B}u + \sum_{n\geq 0, l\geq 2} c_{ln}u^{\mu n+l}$$

Moreover we can show that $m_{ln} = c_{ln} = 0$ for $2 \leq l \leq n$ by induction on l. In fact, fix $n \geq 2$. Then $m_{2n} = c_{2n} = 0$, since, otherwise, dm/du, therefore the right-hand side of (21a), or dr/du, therefore the right-hand side of (21b), would contain the term $u^{\mu n+1}$ with $n \geq 2$, which is impossible. Therefore $m_{2n} = c_{2n} = 0$. Consider $3 \leq l \leq n$. Assume $m_{n',l'} = c_{n',l'} = 0$ for $2 \leq l' \leq n'$ with $n' \leq n, l' \leq l-1$. If $m_{nl} \neq 0$, or, $c_{nl} \neq 0$, then dm/du, or dr/du would contain the term $u^{\mu n+l-1}$, which is impossible by the induction assumption. Therefore $m_{nl} = c_{nl} = 0$ for $2 \leq l \leq n$. This implies that

$$\begin{aligned} r &= r_+ - \frac{1}{B}u + \sum_{n\geq 0, l\geq n+1, l\geq 2} c_{ln}u^{(\mu+1)n+l-n} \\ &= r_+ - \frac{1}{B}u(1 + [u, u^{\mu+1}]_1). \end{aligned}$$

The inverse function $u = u(r)$ then clearly enjoys an expansion of the form

$$u = B(r_+ - r)(1 + [r_+ - r, (r_+ - r)^{\mu+1}]_1).$$

This completes the proof of Theorem 4.

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Appendix

Le us sketch a proof of Lemma 1.

First we assume that μ is not an integer but a rational number, say, $\mu = q/p, p, q \in \mathbb{N}, p \geq 2$ and p, q are relatively prime. Note that a function given by a convergent power series

$$\varphi(x) = \sum \tilde{c}_{ij} x^i (x^\mu)^j, \qquad |\tilde{c}_{ij}| \leq \frac{\tilde{M}}{\tilde{\delta}^{i+j}}, \quad (\tilde{\delta} < 1),$$

can be rewritten as

$$\varphi(x) = \sum_l \sum_{n=0}^{p-1} c_{ln} x^{\mu n + l}, \qquad |c_{ln}| \leq \frac{M}{\delta^l} \quad \text{for} \quad 0 \leq n \leq p-1,$$

where

$$c_{ln} = \sum \{\tilde{c}_{ij} \mid i + qJ = l, j = pJ + n, \exists J \in \mathbb{N}\},$$

and $M = \tilde{M}/(e\tilde{\delta}^p), \delta = \tilde{\delta}/e$. This rewriting is necessary, since not $\tilde{c}_{ij}$'s but c_{ln}'s can be uniquely determined for the given function $\varphi(x)$.

In fact first we note that μn cannot be an integer for $n = 1, \cdots, p-1$. (Proof: Let us deduce a contradiction supposing that nq/p is an integer. We can assume $q < p$, by, if necessary, replacing q by $q' := q - [q/p]p$. Since $nq/p < n$, we see that nq/p is either $1, \cdots$, or $n-1$, therefore q/p is either $1/n, \cdots$, or $(n-1)/n$. Hence p is a divisor of n, a fortiori, $p \leq n$, a contradiction to $n \leq p-1$, QED.) Hence $\mu n + l = \mu n' + l', n, n', l, l' \in \mathbb{N}, 0 \leq n, n' \leq p-1$, implies $n = n', l = l'$. Then we have a unique numbering $(n_k, l_k)_{k \in \mathbb{N}}$ of (n, l)'s such that $\mu n_k + l_k < \mu n_{k+1} + l_{k+1}$. By induction on k we can deduce $c_{l_k n_k} = 0$ for $\forall k$ from $\sum c_{ln} x^{\mu n + l} = 0 \ \forall x$. This means the uniqueness of the coefficients c_{ln} in the above expansion of $\varphi(x)$.

Anyway suppose

$$f^\alpha(x, x^\mu, y_1, y_2) = \sum \sum_{n=0}^{p-1} a^\alpha_{lnk_1k_2} x^{\mu n + l} y_1^{k_1} y_2^{k_2},$$

with

$$\left|a^{\alpha}_{lnk_1k_2}\right| \leq \frac{M}{\delta^{l+k_1+k_2}} \qquad (0 \leq n \leq p-1)$$

and put

$$\begin{aligned} F(x, y_1, y_2) &= \sum \frac{M}{\delta^{l+k_1+k_2}} \sum_{n=0}^{p-1} x^{\mu n+l} y_1^{k_1} y_2^{k_2} \\ &= \frac{M}{1-x/\delta} \frac{1-x^{\mu p}}{1-x^{\mu}} \frac{1}{1-y_1/\delta} \frac{1}{1-y_2/\delta}. \end{aligned}$$

Then the problem

$$\frac{dY}{dx} = F(x, Y, Y), \qquad Y|_{x=0} = 0$$

has a solution of the form

$$\begin{aligned} Y &= Mx(1 + [x, x^{\mu}]_1) \\ &= \sum_l \sum_{n=0}^{p-1} C_{ln} x^{\mu n+l}, \qquad 0 \leq C_{ln} \leq \frac{M'}{(\delta')^l}. \end{aligned}$$

On the other hand (22) has a formal power series solution

$$y_{\alpha} = \sum_l \sum_{n=0}^{p-1} c^{\alpha}_{ln} x^{\mu n+l},$$

where the coefficients c^{α}_{ln}'s are determined by a recursive formula

$$\begin{aligned} c^{\alpha}_{0n} &= 0, \qquad c^{\alpha}_{l+1,n} = \frac{1}{l+1+\mu n} b^{\alpha}_{ln}, \\ b^{\alpha}_{LR} &= \sum a^{\alpha}_{lnk_1k_2} c^{1}_{l'(1)n'(1)} \cdots c^{1}_{l'(k_1)n'(k_1)} c^{2}_{l''(1)n''(1)} \cdots c^{2}_{l''(k_2)n''(k_2)}. \end{aligned}$$

Here the summation in the definition of b^{α}_{LR} is taken over

$$L = qJ + l + l'(1) + \cdots + l'(k_1) + l''(1) + \cdots + l''(k_2)$$

with $l'(1), \cdots, l''(1), \cdots \geq 1$ and $J \in \mathbb{N}$, and

$$pJ + R = n + n'(1) + \cdots + n'(k_1) + n''(1) + \cdots + n''(k_2).$$

Then it can be shown inductively that $|c^{\alpha}_{ln}| \leq C_{ln}$, which implies the convergence of the formal power series solution. This completes the proof.

A proof by a similar and easier majorant argument can be done when μ is an irrational number. Let us omit the repetition.

References

[1] C. G. Böhmer, General relativistic static fluid solutions with cosmological constant: Diplomarbeit, Golm, 16. Juli 2002; arXiv:gr-qc/0308057.

[2] C. G. Böhmer, Static perfect fluid balls with given equation of state and cosmological constant, Ukr. J. Phys., 50(2005), 1219-1225.

[3] P. R. Brady, C. M. Chambers, W. G. Laarakkers and E. Poisson, Radiative faroff in Schwarzschild-de Sitter spacetime, Phys. Rev., D60(1999), 064003.

[4] S. M. Carroll, The cosmological constant, Living Rev. Relativity, 4(2001), 1.

[5] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Univ. Chicago Press, 1939.

[6] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

[7] A. Einstein, Kosmologische Betrachungen zur allgemeinen Relativitätstheorie, Sitzungsberichte der Preussischen Akademie der Wissenschaften, VI(1917), 142-152.

[8] A. Einstein, Zum kosmologischen Problem der allgemeinen Relativitätstheorie, Sitzungsberichte der Preussischen Akademie der Wissenschaften, XII(1931), 235-237.

[9] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problem driven by positive sources, Arch. Rational Mech. Anal., 49(1972/73), 241-269.

[10] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (4th Ed.), Pergamon Press, Oxford, 1975 ; Teorija Polja, Nauka, Moskva, 1973.

[11] T. Makino, On spherically symmetric stellar models in general relativity, J. Math. Kyoto Univ., 38-1(1998), 55-69.

[12] T. Makino, On spherically symmetric solutions of the Einstein-Euler equations, to appear in Kyoto J. Math.; arXiv:1410.1234.

[13] J. P. Oppenheimer and G. M. Volkoff, On massive neutron cores, Phys. Rev., 55(1939), 374-381.

[14] A. Pais, 'Subtle is the Lord ...' The Science and the Life of Albert Einstein, Oxford University Press, 1982.

[15] A. D. Rendall and B. G. Schmidt, Existence and properties of spherically symmetric static fluid bodies with a given equation of state, Class. Quantum Gravity, 8(1991), 985-1000.

[16] W. de Sitter, On the relativity of inertia. Remarks concerning Einstein's latest hypothesis, Proc. Royal Acad. Amsterdam, XIX(1917), 1217-1225.

[17] Ya. B. Zeldovich and I. D. Novikov, Relativistic Astrophysics, Vol. I, Univ. Chicago Press, 1971.

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